Simulating Aggregate Dynamics of Buffer-Stock Savings Models using the Permanent-Income-Neutral Measure*

Karl Harmenberg† Erik Öberg‡

November 16, 2018

Abstract

We propose a method for simulating aggregate dynamics in models where households solve buffer-stock savings problems with permanent income shocks. The idea is to simulate the model using permanent-income neutral probabilities, distorted probabilites which incorporate the effect that permanent income shocks have on macroeconomic aggregates. With permanent-income neutral probabilities, one does not need to keep track of the permanent-income distribution. Our method greatly improves the speed and accuracy of the simulation and can implemented with a few lines of code.

1 Introduction

Buffer-stock savings models are a class of consumption-savings models with an income process that feature both a transitory and a random-walk permanent component. These models are tractable because the household's problem can be normalized by permanent income, thereby reducing the state space when solving for decision functions. However, in a panel of households solving such savings problems, aggregate consumption and savings dynamics depend on the joint distribution of permanent income and normalized asset holdings. Therefore, when simulating aggregate dynamics, one seemingly needs to keep track of the distribution of both permanent income and normalized asset holdings. This is problematic, since with a unit root and perpetual-youth households, the distribution of permanent income has a fat Pareto tail, which gives rise to poor convergence properties. We show that one can simulate the aggregates, only keeping track of the normalized asset distribution, by using what we term the *permanent-income-neutral measure*.

*We are grateful for helpful comments from Jeppe Druedahl. All errors are our own.
†kha.eco@cbs.dk, Copenhagen Business School
‡erik.oberg@nek.uu.se, Uppsala University

What is the problem? Consider a standard buffer-stock savings model with perpetual-youth households (e.g., Carroll et al. (2017) and McKay (2017)). The economy is populated by a continuum of ex-ante identical households. We are interested in macroeconomic aggregates such as total income, total consumption and total savings. The household solves the maximization problem,

$$\mathcal{U} = \max_{\{B_t, C_t\}} E_0 \left[\sum_{t=0}^{\infty} \beta^t \frac{C^{1-\gamma}}{1-\gamma} \right] \text{ s.t. } qB_t + C_t = A_t,$$

$$A_t = B_{t-1} + Y_t,$$

$$B_t \geqslant 0,$$

where income Y_t follows the process

$$Y_t = \epsilon_t Z_t,$$

$$Z_t = \eta_t Z_{t-1}.$$

ϵ' is a transitory income shock ($E[\epsilon'] = 1$) and η' is a permanent income shock ($E[\eta'] = 1$). Households face a constant probability p of dying. When they die, they are replaced by a newborn household with some initial assets A_0 and permanent income $Z = 1$.

The buffer-stock savings model is particularly tractable because the household's problem can be reformulated in terms of normalized assets $a_t = A_t/Z_t$ (Carroll, 1997). The household's state is normalized cash-on-hand $a = A/Z$ and permanent income Z, and the consumption function $C(a_t, Z_t)$ takes the simple form $C(a_t, Z_t) = c(a_t)Z_t$.

We denote the state space $\mathbf{A} \times \mathbf{Z}$, where $\mathbf{A}$ is the normalized cash-on-hand dimension and $\mathbf{Z}$ is the permanent-income dimension. The dynamics in the normalized cash-on-hand dimension $\mathbf{A}$ are well behaved: Convergence to the ergodic steady state is quick and the state space is bounded (if the shock distributions are bounded). The difficulties lie with the dynamics of the permanent-income dimension $\mathbf{Z}$. Convergence to the ergodic steady state in this dimension is slow and accurately estimating total income requires a very large number of households.

As an example, we simulated the evolution of permanent income for 100,000 households over thousand periods using parameter values from Carroll et al. (2017). We repeated the simulation 1,000 times and recorded the mean permanent income of the population. We report the results in the histogram in Figure 1.1. Although the simulations most of the time yield a mean permanent income close to the true value of 1.0, substantial deviations still occur even with 100,000 households. 18.1% of simulations yielded a simulation error large than 0.5% and 1.2% of simulations yielded a simulation error larger than 1.0%.[1] These variations

[1] It is well understood why simulation errors are so large: the random growth model with perpetual-youth households

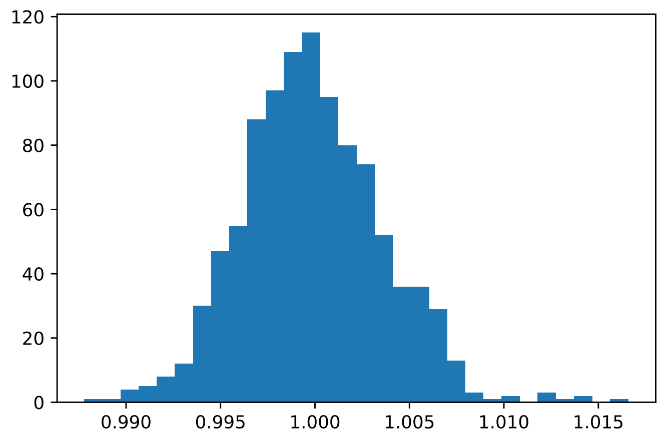

Figure 1.1: The variation in mean permanent income across 1000 simulations, each with 100,000 households. Mean permanent income was computed after 1000 burn-in periods. The simulation used variance $\sigma^2 = 0.0036$ of the permanent shock and per-period death probability $p = 0.00625$, taken from Carroll et al. (2017).

Permanent income Z'	Normalized assets a'	Mass of households	Mass of purchasing power
$1.5Z$	$a'_1 = g(a)/1.5 + 1$	0.5	$0.5 \cdot 1.5 = 0.75$
$0.5Z$	$a'_2 = g(a)/0.5 + 1$	0.5	$0.5 \cdot 0.5 = 0.25$

Table 1.1: The distribution of households in the second period.

are substantial and noticeable, on the same order of magnitude as a business cycle shock.

This simulation imprecision in the permanent-income dimension is a nuisance. For many macroeconomic questions, we do not need to know the full joint distribution of normalized cash on hand *and* permanent income $\mu_{\mathbf{A}\times\mathbf{Z}}$ as long as we know the sufficient statistic for macroeconomic aggregates, the permanent-income weighted cash-on-hand distribution $\tilde{\mu}_{\mathbf{A}}$. The permanent-income weighted cash-on-hand distribution is defined by $\tilde{\mu}_{\mathbf{A}}(\mathcal{A}) = \int_{\mathcal{A}\times\mathbf{Z}} Z d\mu_{\mathbf{A}\times\mathbf{Z}}$ for all measurable sets $\mathcal{A} \subset \mathbf{A}$. For example, total consumption expenditures are given by $\int c(a) d\tilde{\mu}_{\mathbf{A}}$ where c is the normalized consumption function.

Our method improves the speed and accuracy of simulation since it tracks the permanent-income weighted cash-on-hand distribution directly, thereby bypassing the simulation difficulties along the permanent-income dimension.

Intuition Consider the following simple two-period setting. There is a continuum of households at state (a, Z). They choose to save $B = bZ = g(a)Z$, where g is the normalized decision function of the households. Going in to next period, half of the households receive a shock 1.5 to their permanent income and half of

generates a stationary distribution with a Pareto tail (Reed, 2001). The granularity literature (Gabaix, 2011, Carvalho and Grassi, 2019) explores to what extent idiosyncratic shocks at the individual household or firm level can generate aggregate fluctuations, essentially exploiting the poor convergence properties.

Permanent income Z'	Normalized assets a'	Mass of households	Mass of purchasing power
Z	$a'_1 = g(a)/1.5 + 1$	$\tilde{p}$	$\tilde{p}$
Z	$a'_2 = g(a)/0.5 + 1$	$1 - \tilde{p}$	$1 - \tilde{p}$

Table 1.2: The distribution of households in the second period in the alternative model.

the households receive a shock 0.5 to their permanent income. There is no transitory shock.

The distribution of households is given by Table 1.1. Half of the households hold normalized assets $a'_1 = g(a)/1.5 + 1$ and half of the households hold normalized assets $a'_2 = g(a)/0.5 + 1$. The households holding normalized assets a'_1 have a three times as large permanent income as the households holding a'_2, so when we compute macroeconomic aggregates they need to be weighted appropriately. Although half of the households hold normalized assets a'_1, three fourths of the purchasing power is held by households with assets a'_1. Total consumption is given by $\bar{C}' = 0.5 \cdot 1.5Zc(a'_1) + 0.5 \cdot 0.5Zc(a'_2)$.

Now, *formally* consider the following auxilliary environment: Permanent income is always constant at Z but normalized assets are hit by shocks as in the original model, so $a' = g(a)/\eta + 1$. As in the original model, η can be either 1.5 or 0.5. Let $\tilde{p}$ of the households receive shock 1.5 and let $1 - \tilde{p}$ of the households receive the shock 0.5. The distribution of households in the auxilliary model is given by Table 1.2. A share $\tilde{p}$ of the households hold normalized assets a'_1 and a share $1 - \tilde{p}$ hold normalized assets a'_2, similar to the original model. In this auxilliary environment we do not need to weigh the households differently when computing macroeconomic aggregates so $\bar{C}'_{alt} = \tilde{p}c(a'_1) + (1 - \tilde{p})c(a'_2)$.

Note that if we set $\tilde{p} = 0.75$, aggregate consumption is equal in the two model frameworks. In one world, people receive shocks which affects both permanent income and normalized assets, while in the other world people only receive shocks that affect normalized assets. Still, if we pick $\tilde{p}$ appropriately, the two models have the same distribution of purchasing power over the normalized asset dimension and therefore the same aggregate consumption. In fact, all aggregate variables that depend on the distribution of purchasing power over the normalized asset distribution are identical for the two models, the models are observationally equivalent with respect to macroeconomic aggregates.

This example generalizes, as we show in the next section. If $\{\eta_i\}$ are the different possible values of the permanent-income shock and the corresponding probabilities are $\{p_i\}$, then let $\tilde{p}_i = p_i\eta_i$. An auxilliary model where the shock η affects the normalized assets but not permanent income is observationally equivalent to the original model with respect to macroeconomic aggregates, as long as we make the probability adjustment $\tilde{p}_i = p_i\eta_i$. We call $\{\tilde{p}_i\}$ the *permanent-income neutral measure*, in analogy with the risk-neutral measure from asset pricing.

Simulating the aggregate economy The above argument suggests a straight-forward way to simulate the aggregate economy without needing to keep track of the distribution of permanent incomes: Simulate it using the permanent-income-neutral measure. The following is a cookbook recipe for computing aggregate dynamics:

1. Solve for individual behavior g using standard methods.
2. Replace the true probability distribution for η, $\{p_i^\eta\}$, with the permanent-income neutral probabilities $\{\tilde{p}_i\} = \{\eta_i p_i^\eta\}$.
3. Simulate the auxilliary model using standard methods, only keeping track of the normalized asset distibution.
4. Aggregate consumption is then given by the simulated mean normalized consumption, and similarly for savings.

Note that this method bypasses keeping track of the permanent-income distribution, which greatly improves the precision of the simulations.

2 Deriving the law of motion for the permanent-income weighted distribution

We now proceed to formally prove the equivalence between the evolution of aggregates for the original model and the evolution of aggregates for the auxilliary model with permanent-income neutral probabilities.

Environment For expositional purposes, we assume that shocks are discretely distributed (which allows us to easily separate sums belonging to shocks from integrals belonging to the distribution).

- An individual's state is given by cash on hand a and permanent income Z.
- An individual makes decisions which lead to bond holdings $b = g(a)$. We do not need to specify the micro foundations of g, as long as the individual's decision only depends on a and not on Z. We do not assume that g is continuous or invertible, merely that g is measurable.
- The stochastic environment maps b, z to a new state, $(b, Z) \mapsto (b/\eta + \epsilon, Z\eta)$, where η is a shock to permanent income and ϵ is a transitory shock to income.
- The distribution of individuals over states $(a, Z) \in \mathbf{A} \times \mathbf{Z}$ is given by the measure $\mu_{\mathbf{A}\times\mathbf{Z}}$ (where both $\mathbf{A} \subset \mathbb{R}$ and $\mathbf{Z} \subset \mathbb{R}$ are endowed with the Borel σ-algebra).
- The distribution of households over normalized asset is given by $\mu_{\mathbf{A}}(\mathcal{A}) = \int_{\mathcal{A}\times\mathbf{Z}} d\mu_{\mathbf{A}\times\mathbf{Z}}$, where $\mathcal{A}$ is a measurable subset of $\mathbf{A}$. This distribution is not the right distribution for macroeconomic aggregates.

For macroeconomic aggregates, the contribution of a household needs to be weighted by its permanent income.

- For aggregate variables, $\tilde{\mu}_{\mathbf{A}}(\mathcal{A}) = \int_{\mathcal{A}\times\mathbf{Z}} Z d\mu_{\mathbf{A}\times\mathbf{Z}}$ is a sufficient statistic for the distribution. For example, total consumption is given by $C = \int c(a) d\tilde{\mu}_{\mathbf{A}}$.

We now proceed to show how to get a law of motion for $\tilde{\mu}_{\mathbf{A}}$ by using the permanent-income-neutral measure.

For a given pair of shocks η', ϵ' we have $(a, Z) \mapsto (a', Z') = (g(a)/\eta' + \epsilon', \eta' Z)$. Write $p_i^\eta = P(\eta' = \eta_i)$ and $p_j^\epsilon = P(\epsilon' = \epsilon_i)$. The measure of households with state (a', z') in the set $\mathcal{A}' \times \mathcal{Z}'$ is then given by

$$\mu_{\mathbf{A}'\times\mathbf{Z}'}(\mathcal{A}' \times \mathcal{Z}') = \sum_i \sum_j p_i^\eta p_j^\epsilon \mu_{\mathbf{B}\times\mathbf{Z}}((\eta_i(\mathcal{A}' - \epsilon_i)) \times (\mathcal{Z}'/\eta_i)) = \sum_i \sum_j p_i^\eta p_j^\epsilon \mu_{\mathbf{A}\times\mathbf{Z}}(g^{-1}(\eta_i(\mathcal{A}' - \epsilon_i)) \times (\mathcal{Z}'/\eta_i))$$

and therefore the weighted measure $\tilde{\mu}_{\mathbf{A}'}$ is given by

$$\begin{aligned}
\tilde{\mu}_{\mathbf{A}'}(\mathcal{A}') &= \int_{\mathcal{A}'\times\mathbf{Z}'} Z' d\mu_{\mathbf{A}'\times\mathbf{Z}'} = \sum_i \sum_j p_i^\eta p_j^\epsilon \int_{(g^{-1}(\eta_i(\mathcal{A}'-\epsilon_i))\times\mathbf{Z}} (\eta_i Z) d\mu_{\mathbf{A}\times\mathbf{Z}} \\
&= \sum_i \sum_j (p_i^\eta \eta_i) p_j^\epsilon \tilde{\mu}_{\mathbf{A}}(g^{-1}(\eta_i(\mathcal{A}' - \epsilon_i))) = \\
&= \tilde{E}_\eta \left[E_\epsilon[\tilde{\mu}_{\mathbf{A}}(g^{-1}(\eta_i(\mathcal{A}' - \epsilon_i)))] \right]
\end{aligned}$$

where $\tilde{E}_\eta$ is computed using the permanent-income neutral probabilities $\{\tilde{p}_i\} = \{p_i \eta_i\}$.

Similarly, the unweighted measure is given by

$$\begin{aligned}
\mu_{\mathbf{A}'}(\mathcal{A}') &= \int_{\mathcal{A}'\times\mathbf{Z}'} d\mu_{\mathbf{A}'\times\mathbf{Z}'} = \sum_i \sum_j p_i^\eta p_j^\epsilon \int_{(g^{-1}(\eta_i(\mathcal{A}'-\epsilon_i))\times\mathbf{Z}} d\mu_{\mathbf{A}\times\mathbf{Z}} \\
&= \sum_i \sum_j p_i^\eta p_j^\epsilon \mu_{\mathbf{A}}(g^{-1}(\eta_i(\mathcal{A}' - \epsilon_i))) = \\
&= E_\eta \left[E_\epsilon[\mu_{\mathbf{A}}(g^{-1}(\eta_i(\mathcal{A}' - \epsilon_i)))] \right]
\end{aligned}$$

We see that the law of motion for the weighted measure is the same as the law of motion for the unweighted measure, except that the probability distribution for the permanent shock is replaced by the permanent-income-neutral distribution $\{\tilde{p}\} = \{p_i \eta_i\}$.

3 Extensions and remarks

Marginal propensity to consume $\tilde{\mu}_{\mathbf{A}}$ is not a sufficient statistic for all policy-relevant aggregate statistics. One statistic which is not summarized by $\tilde{\mu}_{\mathbf{A}}$ is average marginal propensity to consume. If the government gave 1 dollar to each household, how much (in partial equilibrium) would consumption increase? An individual household increases consumption by $\frac{\partial}{\partial A}[zc(A/z)] = c'(a)$ so average marginal propensity to consume is given by $\overline{MPC} = \int c'(a) d\mu_{\mathbf{A}\times\mathbf{Z}} = \int c'(a) d\mu_{\mathbf{A}}$.

If we want to track average marginal propensity to consume and aggregate consumption, we need to simulate both the evolution of the weighted and the unweighted measure. Note however that we are better off simulating two well-behaved one-dimensional processes than one not well-behaved two-dimensional process (which would be the alternative if tracking the full distribution).

Stochastic death à la Blanchard-Yaari If we introduce stochastic death and birth à la Blanchard-Yaari, we get

$$\tilde{\mu}_{\mathbf{A}'}(\mathcal{A}') = (1-\lambda)\left\{\tilde{E}_\eta\left[E_\epsilon[\tilde{\mu}_{\mathbf{A}}(g^{-1}(\eta_i(\mathcal{A}'-\epsilon_i)))]\right]\right\} + \lambda\tilde{\nu}(A')$$

where $\tilde{\nu}$ is the permanent-income-weighted asset distribution of the newborn.

Extension with durable goods and nonconvexities Consider a setup with both a liquid asset and a durable good subject to nonconvex adjustment costs (as in Harmenberg and Öberg (2018)). By simply reinterpreting a and b as vectors, the logic of the argument goes through. Note that in this case, g, the mapping of assets to saving decisions, is neither injective nor continuous, but as long as the mapping is measurable the logic still holds.

Tracking variances The method can be adopted to track variances of e.g. consumption as well. Define $\tilde{\tilde{\mu}}_{\mathbf{A}}(\mathcal{A}) = \int_{\mathcal{A}\times\mathbf{Z}} Z^2 d\mu_{\mathbf{A}\times\mathbf{Z}}$. Then the evolution of $\tilde{\tilde{\mu}}_{\mathbf{A}}$ is given by

$$\tilde{\tilde{\mu}}_{\mathbf{A}'}(\mathcal{A}') = (1-\lambda)E[(\eta')^2]\left\{\tilde{\tilde{E}}_\eta[E_\epsilon[\tilde{\tilde{\mu}}_{\mathbf{A}}(g^{-1}(\eta_i(\mathcal{A}'-\epsilon_i)))]]\right\} + \lambda\tilde{\tilde{\nu}}(\mathcal{A}')$$

where $\tilde{\tilde{p}}_i = p_i\eta_i^2/E[(\eta')^2]$ and the $\tilde{\tilde{E}}[\cdot]$ expectation is taken using probabilities $\{\tilde{\tilde{p}}_i\}$. The variance of consumption is then given by $\int c(a)^2 d\tilde{\tilde{\mu}}_{\mathbf{A}'} - \left(\int c(a) d\tilde{\mu}_{\mathbf{A}'}\right)^2$. Similarly, the method can be adopted to track higher-order moments.

Continuous-time model In continuous time, the dynamics take a particular and simple form. The model law of motion is given by

$$dA_t = (Z_t - c(a_t)Z_t + rA_t)dt$$
$$dZ_t = \sigma Z_t dW_t$$

which after a change of variable $a_t = A_t/Z_t$ and applying Itô's lemma gives

$$
\begin{aligned}
da_t &= (1 - c(a_t) + (r + \sigma^2/2)a_t)dt - \sigma a_t dW_t, \\
dZ_t &= \sigma Z_t dW_t.
\end{aligned}
$$

Households die at rate λ and are replaced by households drawn from the distribution $g(a, Z)$.

The Kolmogorov forward equation for the weighted distribution $\tilde{f}$ is given by

$$
\begin{aligned}
\frac{\partial \tilde{f}}{\partial a}(a) =& - \frac{\partial}{\partial a}[(1 - c(a) + (r - \sigma^2/2)a)\tilde{f}(a)] + \frac{1}{2}\frac{\partial^2}{\partial a^2}[\sigma^2 a^2 \tilde{f}(a)] \\
& - \lambda \tilde{f}(a) + \lambda \tilde{g}(a),
\end{aligned}
$$

and the Kolmogorov forward equation for the unweighted distribution f is given by

$$
\begin{aligned}
\frac{\partial f}{\partial a}(a) =& - \frac{\partial}{\partial a}[(1 - c(a) + (r + \sigma^2/2)a)f(a)] + \frac{1}{2}\frac{\partial^2}{\partial a^2}[\sigma^2 a^2 f(a)] \\
& - \lambda f(a) + \lambda g(a).
\end{aligned}
$$

The law of motion for the weighted distribution is identical to the law of motion for the unweighted distribution with the interest rate r replaced by $r - \sigma^2$. For details, see Appendix A.

4 Conclusion

In this note, we introduced a simple way to track the evolution of the permanent-income weighted distribution without tracking the permanent-income distribution directly. Replace actual probabilities with permanent-income neutral probabilities in the simulation. This adjustment greatly improves computational efficiency.

References

Carroll, C. (1997). Buffer-Stock Saving and the Life Cycle/Permanent Income Hypothesis. *Quarterly Journal of Economics*, 112(1):1–55.

Carroll, C., Slacalek, J., Tokuoka, K., and White, M. N. (2017). The distribution of wealth and the marginal propensity to consume. *Quantitative Economics*, 8(3):977–1020.

Carvalho, V. M. and Grassi, B. (2019). Large Firm Dynamics and the Business Cycle. *American Economic Review*, Forthcomin.

Gabaix, X. (2011). The Granular Origins of Aggregate Fluctuations. *Econometrica*, 79(3):733–772.

Harmenberg, K. and Öberg, E. (2018). Consumption Dynamics under Time-Varying Unemployment Risk. *Mimeo.*

McKay, A. (2017). Time-varying idiosyncratic risk and aggregate consumption dynamics. *Journal of Monetary Economics*, 88:1–14.

Reed, W. J. (2001). The Pareto, Zipf and other power laws. *Economics Letters*, 74(1):15–19.

A Appendix

First, we note that the Kolmogorov forward equation for the unweighted distribution is given by

$$\begin{aligned}\frac{\partial f}{\partial a}(a) =& -\frac{\partial}{\partial a}[(1-c(a)+(r+\sigma^2/2)a)f(a)] + \frac{1}{2}\frac{\partial^2}{\partial a^2}[\sigma^2 a^2 f(a)] \\ &-\lambda f(a) + \lambda g(a).\end{aligned}$$

Next, we set out to compute the Kolmogorov forward equation for the permanent-income weighted distribution. Denote by $f_t(a, Z)$ the distribution of agents over the state space. We allow for drift $\mu \neq 0$ in permanent income. The multivariate Kolmogorov forward equation gives

$$\begin{aligned}\frac{\partial f}{\partial t}(a, Z) =& -\frac{\partial}{\partial a}[(1-c(a)+(r+\sigma^2/2-\mu)a)f(a, Z)] + \frac{1}{2}\frac{\partial^2}{\partial a^2}[\sigma^2 a^2 f(a, Z)] \\ &-\frac{\partial}{\partial Z}[(\mu Z)f(a, Z)] + \frac{1}{2}\frac{\partial^2}{\partial Z^2}[\sigma^2 Z^2 f(a, Z)] \\ &-\frac{\partial^2}{\partial a \partial Z}[\sigma^2 a Z f(a, Z)] - \lambda f(a, Z) + \lambda g(a, Z)\end{aligned}$$

where λ is a death probability and g is the distribution for newborns.

For aggregate dynamics, $\tilde{f}(a) := \int f(a, Z) Z dZ$ is a sufficient statistic. The ultimate goal of this section is to compute a law of motion for $\tilde{f}(a)$.

Integrating over the Kolmogorov forward equation, we get

$$\begin{aligned}\frac{\partial \tilde{f}}{\partial a}(a) =& -\frac{\partial}{\partial a}[(1-c(a)+(r+\sigma^2/2-\mu)a)\tilde{f}(a)] + \frac{1}{2}\frac{\partial^2}{\partial a^2}[\sigma^2 a^2 \tilde{f}(a)] \\ &-\int \frac{\partial}{\partial Z}[(\mu Z)f(a, Z)] Z dZ + \frac{1}{2}\int \frac{\partial^2}{\partial Z^2}[\sigma^2 Z^2 f(a, Z)] Z dZ \\ &-\int \frac{\partial^2}{\partial a \partial Z}[\sigma^2 a Z f(a, Z)] Z dZ - \lambda \tilde{f}(a) + \lambda \tilde{g}(a).\end{aligned}$$

Next, we evaluate the three integrals by repeated application of integration by parts.

Integral 1

$$\int \frac{\partial}{\partial Z}[(\mu Z)f(a,Z)]ZdZ = \mu[Z^2 f(a,Z)]_0^\infty - \mu \int Zf(a,Z)dZ = -\mu\tilde{f}(z)$$

under the assumptions that $\lim_{Z\to 0} Z^2 f(a,Z) = 0$ and $\lim_{Z\to\infty} Z^2 f(a,Z) = 0$. The first condition is satisfied as long as f is bounded in a neighborhood around 0 and the second condition is satisfied as long as the tail index α is greater than 1 for all a.

Integral 2

$$\begin{aligned}\int \frac{\partial^2}{\partial Z^2}[\sigma^2 Z^2 f(a,Z)]ZdZ =& \sigma^2 \left[\frac{\partial}{\partial Z}[Z^2 f(a,Z)]Z\right]_0^\infty - \sigma^2 \int \frac{\partial}{\partial Z}[Z^2 f(a,Z)]dZ \\ =& \sigma^2 [Z^2 f(a,Z) + Z^3 \frac{\partial}{\partial Z} f(a,Z)] = 0\end{aligned}$$

under the assumption that the tail index α is larger than 1.

Integral 3

$$\int \frac{\partial^2}{\partial a \partial Z}[\sigma^2 a Z f(a,Z)]ZdZ = \sigma^2 \frac{\partial}{\partial a}\left\{a \int \frac{\partial}{\partial Z}[Zf(a,Z)]ZdZ\right\} = -\sigma^2 \frac{\partial}{\partial a}[a\tilde{f}(a)]$$

where the integral inside the curly bracket is analogous with integral 1.

Putting it all together, we get

$$\begin{aligned}\frac{\partial \tilde{f}}{\partial a}(a) =& -\frac{\partial}{\partial a}[(1 - c(a) + (r + \sigma^2/2 - \mu)a)\tilde{f}(a)] + \frac{1}{2}\frac{\partial^2}{\partial a^2}[\sigma^2 a^2 \tilde{f}(a)] \\ &+ \mu\tilde{f} + \sigma^2 \frac{\partial}{\partial a}[a\tilde{f}(a)] - \lambda\tilde{f}(a) + \lambda\tilde{g}(a).\end{aligned}$$